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Quantum automorphism groups of connected locally finite graphs and quantizations of finitely generated groups

Quantum Groups Seminar

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Quantum automorphism groups of finite graphs

Definition (Bichon 1999, Banica 2003)

The **quantum automorphism group** $\mathrm{QAut}\,\Pi$ of a finite graph Π with vertex set I is the compact quantum group generated by an $I \times I$ unitary representation $(u_{ij})_{i,j\in I}$ satisfying

- $u_{ij} = u_{ij}^* = u_{ij}^2$ and $\sum_{k\in I} u_{ik} = 1 = \sum_{k\in I} u_{kj}$ $\rightsquigarrow$ magic unitary U
- $\sum_{k:k\sim j} u_{ik} = \sum_{k:k\sim i} u_{kj}$ $\rightsquigarrow$ $UA = AU$, where A is the adjacency matrix

- Closed quantum subgroup of S_n^+ with $n = |I|$. Not necessarily a finite quantum group.
- The map $u_{ij} \mapsto \text{indicator}\{\sigma \in \mathrm{Aut}\,\Pi \mid i = \sigma(j)\}$ turns $\mathrm{Aut}\,\Pi$ in a closed quantum subgroup of $\mathrm{QAut}\,\Pi$.

A rich source of compact quantum groups

Schmidt 2019: graphs with disjoint automorphisms have quantum symmetry

Let Π be a finite graph with vertex set $I = I_1 \sqcup I_2$. Denote by G_k the group of automorphisms σ s.t. $\sigma(i) = i$ for all $i \in I_k$.

- If both G_1 and G_2 are nontrivial, then Π has quantum symmetry: $\operatorname{Aut}\Pi \neq \operatorname{QAut}\Pi$.
- If moreover $|G_1| + |G_2| \geq 5$, then $\operatorname{QAut}\Pi$ is not even co-amenable.

Theorem (combining Kuperberg 1996, Arano 2014 and Edge 2019)

If Π is the Higman-Sims graph, then $\mathbb{G} = \operatorname{QAut}\Pi$ is monoidally equivalent with $\mathrm{SO}_q(5)$.

- $\widehat{\mathbb{G}}$ has property (T) and is infinite,
- but the fusion rules of $\mathbb{G}$ are abelian.

Quantum automorphism groups of infinite graphs

Main goal: construct $\text{QAut}\,\Pi$ for **infinite** graphs Π, as a **locally compact** quantum group.

- The classical $\text{Aut}\,\Pi$ is a topological group for the topology of pointwise convergence.
- In this way, $\text{Aut}\,\Pi$ is only a Polish group.
- If Π is connected and locally finite, then $\text{Aut}\,\Pi$ is a locally compact group.

More precise goal: construct the locally compact quantum group $\text{QAut}\,\Pi$ for connected locally finite graphs Π.

$\rightsquigarrow$ Joint work with Lukas Rollier

$\rightsquigarrow$ A rich class of new examples of locally compact quantum groups.

Example: $\text{QAut}(d\text{-regular tree})$

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Some graph theory conventions

A **graph** is a pair (I, E) of a **vertex set** I and a subset $E \subset I \times I$ of **edges** such that $(i, j) \in E$ whenever $(j, i) \in E$.

- No orientation, no multiple edges.
- Loops are allowed.

⤳ The **automorphism group** $\operatorname{Aut} \Pi$ of a graph $\Pi = (I, E)$ consists of all permutations $\sigma : I \to I$ such that $(\sigma \times \sigma)(E) = E$.

- A **path** $(i_0, \ldots, i_n)$ is a sequence of vertices with $i_{k-1} \sim i_k$ ⤳ **connectedness**
- **Degree** $\deg i = \#\{j \mid j \sim i\}$ ⤳ **local finiteness**

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A multiplier Hopf $*$-algebra

Let Π be a connected locally finite graph with vertex set I.

The underlying $*$-algebra

There is a unique universal $*$-algebra $\mathcal{A}$ with generators $(u_{ij})_{i,j\in I}$ satisfying:

- $u_{ij}^* = u_{ij} = u_{ij}^2$. If $i \neq j$, then $u_{ik}u_{jk} = 0 = u_{ki}u_{kj}$.
- $\sum_{k\in I} u_{ik} = 1 = \sum_{k\in J} u_{kj}$ strictly, and $\sum_{k:k\sim j} u_{ik} = \sum_{k:k\sim i} u_{kj}$

There is a unique nondegenerate $*$-hom $\Delta : \mathcal{A} \to M(\mathcal{A}\otimes\mathcal{A}) : \Delta(u_{ij}) = \sum_{k\in I} u_{ik}\otimes u_{kj}$ strictly.

⇝ **Easy:** the pair $(\mathcal{A}, \Delta)$ is a **multiplier Hopf $*$-algebra** in the sense of Van Daele.

⇝ **Difficult:** existence of the Haar measure.

Existence of the Haar measure

Theorem (Rollier-V 2022)

The multiplier Hopf $*$-algebra $(\mathcal{A}, \Delta)$ associated with a connected locally finite graph Π admits a positive faithful left invariant functional φ and a positive faithful right invariant functional ψ.

- This means that $(\mathcal{A}, \Delta)$ is an **algebraic quantum group** in the sense of Kustermans and Van Daele.
- By their work: the C*-algebra and von Neumann algebra completion in the GNS-construction of φ automatically gives a **locally compact quantum group**.
- Notation: $\mathrm{QAut}\,\Pi$. Note that $\mathrm{Aut}\,\Pi$ is a closed quantum subgroup of $\mathrm{QAut}\,\Pi$.
- We have $S^2 = \mathrm{id}$. The functionals φ and ψ are tracial, but need not be equal (non-unimodularity).

A unitary tensor category approach

A construction of Arano-V, 2016

Let G be a locally compact group and $G \curvearrowright I$ a transitive action on a countable set.

Consider the $\ell^\infty(I)$-Hilbert-bimodules $\mathcal{H}$ that are

- **of finite type**: $p_i \cdot \mathcal{H}$ and $\mathcal{H} \cdot p_i$ have finite dim for every minimal projection $p_i \in \ell^\infty(I)$,
- ***G*-equivariant**: unitary rep $\pi : G \to \mathcal{U}(\mathcal{H})$ such that $\pi(g)(p_i \cdot \mathcal{H} \cdot p_j) = p_{g \cdot i} \cdot \mathcal{H} \cdot p_{g \cdot j}$.

Together with $\mathcal{H} \otimes_I \mathcal{K} = \bigoplus_{i \in I} (\mathcal{H} \cdot p_i \otimes p_i \cdot \mathcal{K})$, we get a unitary tensor category $\mathcal{C}(G \curvearrowright I)$.

Theorem (Arano-V, 2016)

For $I = G/K$ with $K \subset G$ a compact open subgroup, the Drinfeld center of $\mathcal{C}(G \curvearrowright I)$ and the quantum double of G are equivalent.

Approach to construct the Haar measure on $\mathrm{QAut}\,\Pi$

Let Π be a connected locally finite graph with vertex set I.

- We first define a unitary tensor category $\mathcal{C}(\Pi)$.
- We **use this as a tool** to construct the Haar functionals on the multiplier Hopf $*$-algebra $(\mathcal{A}, \Delta)$.

  So we get the locally compact quantum group $\mathrm{QAut}\,\Pi$.
- We then show that $\mathcal{C}(\Pi)$ is equivalent with the unitary tensor category of $\mathrm{QAut}\,\Pi$-equivariant $\ell^\infty(I)$-bimodules of finite type.
- For simplicity, we are assuming here **(quantum) vertex transitivity**.

  In general: a unitary 2-category where the 0-cells are the quantum orbits $I_a \subset I$.

The Mancinska-Roberson approach to QAut Π

Let Π be a **finite** graph with vertex set I.

- We have the fundamental representation U on $\ell^2(I)$.
- Consider $\mathrm{Mor}(n,m) \subset M_{I^n \times I^m}(\mathbb{C})$, the space of morphisms between the m-fold and n-fold tensor power of U.

Theorem (Mancinska-Roberson, 2019)

We have that $\mathrm{Mor}(n,m)$ equals the linear span of the $I^n \times I^m$ matrices $T^{\mathcal{K}}$, where

- $\mathcal{K} = (K, x, y) \in \mathcal{P}(n,m)$ is a **planar bi-labeled graph**, $x \in V(K)^n$ and $y \in V(K)^m$,
- $T^{\mathcal{K}}_{ij} = \#\{\varphi : V(K) \to I \mid \varphi \text{ is a graph homomorphism, and } \varphi(x) = i, \varphi(y) = j\}$.

⇝ We consider a variant of this, to define a unitary tensor category of $\ell^\infty(I)$-bimodules of finite type.

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A unitary tensor category of a connected locally finite graph Π

Let Π be a connected locally finite graph with vertex set I. Assume vertex transitivity.

Notations

- $\mathcal{L}(n, m)$ is the set of all connected planar bi-labeled graphs $(K, x, y) \in \mathcal{P}(n+1, m+1)$ with $x_0 = y_0$ and $x_n = y_m$.
- $\mathrm{Mor}(n, m)$ is the linear span of all $I^{n+1} \times I^{m+1}$ matrices $T^{\mathcal{K}}$ with $\mathcal{K} \in \mathcal{L}(n, m)$.

All $T \in \mathrm{Mor}(n, m)$ define bounded $\ell^\infty(I)$-bimodular operators from $\ell^2(I^{m+1})$ to $\ell^2(I^{n+1})$.

Theorem (Rollier-V, 2022)

We can define a unitary tensor category $\mathcal{C}(\Pi)$ in which

- the objects are (finite direct sums of) projections in $\mathrm{Mor}(n, n)$, $n \in \mathbb{N}$.
- the $P \, \mathrm{Mor}(n, m) Q$ are the morphism spaces, and $P \otimes_I Q$ is the tensor product.

A formula for the Haar measure on QAut Π

Let Π be a connected locally finite graph with vertex set I. Assume vertex transitivity.

Theorem (Rollier-V, 2022)

Fix a base vertex $e \in I$. The formula $\varphi_e(u_{i_1 j_1} \cdots u_{i_n j_n}) = \sum_{k \in I} \sum_{V \in \mathrm{onb}} V_{kik,k} \overline{V_{eje,e}}$

where onb is an orthonormal basis of isometries in $\mathrm{Mor}(n+1, 0)$ defines a faithful, positive, left invariant functional on the multiplier Hopf $*$-algebra $(\mathcal{A}, \Delta)$.

- There is a (unique up to multiple) function $\mu : I \to (0, +\infty)$ such that $\varphi_e(u_{ij}) = \mu_j \mu_e^{-1}$ and $\mu_f \, \varphi_f = \mu_e \, \varphi_e$.
- The modular element is $\delta = \sum_{i \in I} \mu_i \mu_j^{-1} \, u_{ij}$, independently of $j \in I$.
- Without vertex transitivity: first define quantum orbits $I_a \subset I$; then a unitary 2-category $\mathcal{C}(\Pi)$; then φ_e; and finally $u_{ij} \neq 0$ iff i and j lie in the same quantum orbit.

Quantum automorphism groups of Cayley graphs

Let Γ be a discrete group with finite symmetric generating set $S = S^{-1} \subset \Gamma$.

⟿ The Cayley graph Π has vertex set Γ and edge set $\{(g,h) \in \Gamma \times \Gamma \mid g^{-1}h \in S\}$.

⟿ Using V-Valvekens (2018), the unitary tensor category $\mathcal{C}(\Pi)$ has a canonical fiber functor.

Theorem (Rollier-V, 2022)

The resulting compact quantum group $\mathbb{G}$ is the universal compact quantum group generated by an $S \times S$ unitary representation U such that for every $n \geq 2$, the following vector $\xi_n \in \ell^2(S^n)$ is invariant under the n-fold tensor power of U.

$$\xi_n(s_1, \ldots, s_n) = 1 \quad \text{if} \quad s_1 \cdots s_n = e \quad \text{and} \quad \xi_n(s_1, \ldots, s_n) = 0 \quad \text{otherwise.}$$

⟿ Canonical surjective $\pi : \mathcal{O}(\mathbb{G}) \to \mathbb{C}[\Gamma] : \pi(U_{st}) = \delta_{s,t}\, s$.

Quantizations of discrete groups

Definition (Rollier-V, 2022)

Let $\mathbb{G}$ be a compact quantum group and Γ a discrete group.

When we are given a surjective Hopf $*$-algebra homomorphism $\pi : \mathcal{O}(\mathbb{G}) \to \mathbb{C}[\Gamma]$ that does not admit a nontrivial intermediate $\mathcal{O}(\mathbb{G}) \to \mathbb{C}[\Lambda] \to \mathbb{C}[\Gamma]$, we call $\widehat{\mathbb{G}}$ a **quantization** of Γ.

$\rightsquigarrow$ We call the universal compact quantum group associated with (Γ, S) the **planar quantization** of (Γ, S).

- Natural quantizations $\mathcal{O}(A_u(d)) \to \mathbb{C}[\mathbb{F}_d]$ and $\mathcal{O}(A_0(d)) \to \mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^{*d}]$.
- The planar quantization of $(\mathbb{Z}/2\mathbb{Z})^{*d}$ w.r.t. the canonical generators is the dual of the hyperoctohedral quantum group $H^+(d)$.
- The planar quantization of $\mathbb{F}_d$ w.r.t. the canonical generators is the dual of a **twisted** hyperoctohedral quantum group $H_J^+(2d)$.

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“Single” quantum automorphisms of a graph

In a graph Π with vertex set I, write $\mathrm{rel}(i,j) = 1$ if $i \sim j$ and $\mathrm{rel}(i,j) = 0$ if $i \not\sim j$.

Definition (Voigt, 2022)

A **quantum automorphism** of a graph Π with vertex set I consists of a magic unitary $U = (u_{ij})_{i,j\in I}$ on a Hilbert space $\mathcal{H}$ satisfying $u_{ij}\, u_{kl} = 0$ if $\mathrm{rel}(i,k) \neq \mathrm{rel}(j,l)$.

If Π is connected and locally finite, we have the algebraic quantum group $(\mathcal{A}, \Delta)$.

$\rightsquigarrow$ By definition, there is a bijective correspondence between quantum automorphisms of Π and nondegenerate $*$-representations of the $*$-algebra $\mathcal{A}$.

Definition (Rollier-V, 2022)

Two connected locally finite graphs Π and Π' with vertex sets I and I' are said to be **quantum isomorphic** if there exists a magic unitary $V = (v_{ij})_{i\in I, j\in I'}$ on a Hilbert space $\mathcal{H}$ satisfying $v_{ij}\, v_{kl} = 0$ if $\mathrm{rel}(i,k) \neq \mathrm{rel}(j,l)$.

Equivalent characterizations of quantum isomorphic graphs

We generalize results of Brannan$-\cdots-$Wasilewski (2018) and Mancinska-Roberson (2019).

Theorem (Rollier-V, 2022)

For connected locally finite graphs Π and Π', the following are equivalent.

- Π and Π' are **quantum isomorphic**.
- Π and Π' are **algebraically quantum isomorphic**: there exists a nonzero $*$-algebra $\mathcal{B}$ generated by the entries of a "magic unitary" $(v_{ij})_{i\in I, j\in I'}$ satisfying $v_{ij}\, v_{kl} = 0$ if $\mathrm{rel}(i,k) \neq \mathrm{rel}(j,l)$.
- Π and Π' are **planar isomorphic**: the number of pointed homomorphisms from any finite planar graph K to Π and Π' are equal.

Applications

Using two finite graphs that are quantum isomorphic without being isomorphic, and performing a product graph construction, we obtain:

- Examples where $\mathrm{Aut}\,\Pi$ is compact, but $\mathrm{QAut}\,\Pi$ is noncompact and quantum vertex transitive.
- Examples where $\mathrm{Aut}\,\Pi$ is compact, but $\mathrm{QAut}\,\Pi$ is nonunimodular and quantum vertex transitive.
- Examples where $\mathrm{Aut}\,\Pi$ is nonunimodular, but $\mathrm{QAut}\,\Pi$ is unimodular and quantum vertex transitive.